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Non-existence of f-symbols in generalized Taub–NUT spacetimes

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Abstract
In a previous paper it was proved that the extensions of the Taub–NUT geometry do not admit Killing–Yano tensors, even if they possess Stäckel–Killing tensors. Here the analysis is taken further and it is shown that, in general, this class of metrics does not even admit f-symbols. The only exception is the original Taub–NUT metric which possesses four Killing–Yano tensors of valence two.

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1. Extended Taub–NUT spaces

The Euclidean Taub–NUT metric is involved in many modern studies in physics [1, 2]. From the viewpoint of dynamical systems, the geodesic motion in Taub–NUT metric is known to admit a Kepler-type symmetry [3–6]. One can actually find the so-called Runge–Lenz vector as a conserved vector in addition to the angular momentum vector. As a consequence, all the bounded trajectories are closed and the Poisson brackets among the conserved vectors give rise to the same Lie algebra as the Kepler problem has, depending on the energy. Thus the Taub–NUT metric provides a non-trivial generalization of the Kepler problem.

Iwai and Katayama [7] generalized the Taub–NUT metric so that it still admits a Kepler-type symmetry. The extended Taub–NUT metric, denoted by $\mathrm{d}s_{\mathrm{K}}^2$, is defined on $\mathbb{R}^4 - \{0\}$ by

$$\mathrm{d}s_{\mathrm{K}}^2 = f(r)(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2) + g(r)(\mathrm{d}\chi + \cos\theta\,\mathrm{d}\varphi)^2 \tag{1}$$

where $r > 0$ is the radial coordinate, the angle variables (θ, φ, χ) parametrize the unit sphere S^3 with $0 \leqslant \theta < \pi$, $0 \leqslant \varphi < 2\pi$, $0 \leqslant \chi < 4\pi$, and $f(r)$ and $g(r)$ are functions given, with constants a, b, c, d, by

$$f(r) = \frac{a + br}{r} \qquad g(r) = \frac{ar + br^2}{1 + cr + dr^2}. \tag{2}$$

If one takes the constants $c = \frac{2b}{a}$, $d = \frac{b}{a}^2$ with $4m = \frac{a}{b}$, the extended Taub–NUT metric becomes the original Euclidean Taub–NUT metric up to a constant factor.

Spaces with a metric of the above form have an isometry group $SU(2) \times U(1)$. The four Killing vectors are [3]

$$D_A = R_A^\mu \partial_\mu \qquad A = 0, 1, 2, 3 \quad \mu = (r, \theta, \varphi, \chi) \tag{3}$$

with

$$\begin{aligned} R_0 &= (0, 0, 0, 1) \\ R_1 &= (0, -\sin\varphi, -\cot\theta\cos\varphi, \csc\theta\cos\varphi) \\ R_2 &= (0, \cos\varphi, -\cot\theta\sin\varphi, \csc\theta\sin\varphi) \\ R_3 &= (0, 0, 1, 0). \end{aligned} \tag{4}$$

D_0, which generates the $U(1)$ of χ translations, commutes with the other Killing vectors. The remaining three Killing vectors D_i, $i = 1, 2, 3$, correspond to the invariance of the metric (1) under spatial rotations.

The conserved quantities along geodesics are homogeneous functions in momentum p_μ which commute with the Hamiltonian

$$H = \tfrac{1}{2} g^{\mu\nu} p_\mu p_\nu \tag{5}$$

in the sense of Poisson brackets.

The usual constants of motion for particles moving in this background are linear in the four-momentum p_μ:

$$J_A = R_A^\mu \, p_\mu. \tag{6}$$

The conserved quantity corresponding to the cyclic variable χ is given by

$$q = g(r)(\dot\theta + \cos\theta\dot\varphi) \tag{7}$$

where the over-dot denotes an ordinary proper-time derivative. For negative mass models, this quantity can be interpreted as the 'relative electric charge' [2–6]. For $q \in \mathbb{R}$ fixed, the geodesic flow system for the Taub–NUT metric is reduced to a Hamiltonian system on $T^*(\mathbb{R}^3 - \{0\}) \cong (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$ with Cartesian coordinates (x_j, p_j), $j = 1, 2, 3$, and the Hamiltonian function

$$H_q = \frac{\vec{p}^{\,2}}{2f(r)} + \frac{q^2}{2g(r)}. \tag{8}$$

Owing to the obvious spherical symmetry, the angular momentum vector

$$\vec{J} = \vec{x} \times \vec{p} + q\,\frac{\vec{x}}{r} \tag{9}$$

is a conserved vector.

The remarkable result of Iwai and Katayama is that the extended Taub–NUT space (1) still admits a conserved vector, quadratic in four-velocities, analogous to the Runge–Lenz vector of the following form:

$$\vec{S} = \vec{p} \times \vec{J} + \kappa\frac{\vec{x}}{r}. \tag{10}$$

Setting the value of the Hamiltonian H_q to E the constant κ involved in the Runge–Lenz vector (10) is

$$\kappa = -a\,E + \tfrac{1}{2} c\, q^2. \tag{11}$$

The Poisson brackets between the components of $\vec{J}$ and $\vec{S}$ are similar to the relations known for the original Taub–NUT metric [7]. In particular

$$\{J_i, S_j\} = \epsilon_{ijk} S_k. \tag{12}$$

2. Killing–Yano tensors and f-symbols

The explicit form of the 3-vector $\vec{S}$ (10) is

$$\vec{S} = \left[\left(f^2 - \frac{af}{2r}\right)\dot{\vec{x}}^2 + \left(\frac{c}{2r} - \frac{a}{2gr}\right)q^2\right]\vec{x} - \frac{q}{r}\vec{x}\times\vec{p} - f^2 r\dot{r}\dot{\vec{x}}. \tag{13}$$

It is straightforward to verify that the components of the three-vector $\vec{S}$ are Stäckel–Killing tensors of valence two [8]:

$$S_{i(\mu\nu;\lambda)} = 0 \qquad i = 1, 2, 3 \tag{14}$$

confirming the expectations. Actually such integrals of motion are related to hidden symmetries of the manifold, which manifest themselves as Stäckel–Killing tensors.

There are geometries where the Stäckel–Killing tensor can have a certain root represented by Killing–Yano tensors. We recall that a tensor $f_{\mu\nu}$ is a Killing–Yano tensor of valence two if it is totally antisymmetric and satisfies the equation [9]

$$f_{\mu(\nu;\lambda)} = 0. \tag{15}$$

In the original Taub–NUT geometry there are four Killing–Yano tensors [4]. Three of these are covariantly constant

$$\begin{aligned} f_i &= 8m(\mathrm{d}\chi + \cos\theta\,\mathrm{d}\varphi)\wedge\mathrm{d}x_i - \epsilon_{ijk}\left(1 + \frac{4m}{r}\right)\mathrm{d}x_j\wedge\mathrm{d}x_k \\ D_\mu f^{\nu}_{i\lambda} &= 0 \qquad i, j, k = 1, 2, 3. \end{aligned} \tag{16}$$

They are mutually anticommuting and square to minus unity. Thus they are complex structures realizing the quaternion algebra and the original Taub–NUT manifold is hyper-Kähler.

In addition to the above vector-like Killing–Yano tensors there is also a scalar one

$$f_{\mathrm{Y}} = 8m(\mathrm{d}\chi + \cos\theta\,\mathrm{d}\varphi)\wedge\mathrm{d}r + 4r(r + 2m)\left(1 + \frac{r}{4m}\right)\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi \tag{17}$$

which is not covariantly constant.

In the original Taub–NUT space the components $S_{i\mu\nu}$ involved with the Runge-Lenz-type vector (10) can be expressed as symmetrized products of the Killing–Yano tensors f_i (16) and f_{Y} (17) [10]:

$$S_{i\mu\nu} - \frac{1}{8m}(R_{0\mu}R_{i\nu} + R_{0\nu}R_{i\mu}) = m\left(f_{Y\mu\lambda}f_i{}^{\lambda}{}_{\nu} + f_{Y\nu\lambda}f_i{}^{\lambda}{}_{\mu}\right). \tag{18}$$

In fact, only the product of Killing–Yano tensors f_i and f_{Y} leads to non-trivial Stäckel–Killing tensors, the last term in the left-hand side of (18) being a simple product of Killing vectors. This term is usually added to write the Runge–Lenz vector in the standard form (10). The existence of the Killing–Yano tensors is connected with the appearance of additional supersymmetries in the usual $N = 1$ supersymmetric extension of point particle mechanics in curved spacetime.

However, in general, the Stäckel–Killing tensors involved in the Runge–Lenz vector cannot be expressed as a product of Killing–Yano tensors [8]. The extensions of the Taub–NUT geometry do not admit a Killing–Yano tensor, even if they possess Stäckel–Killing tensors. Therefore, the only exception is the original Taub–NUT metric.

If the Killing–Yano tensors are missing, to take up the question of the existence of extra supersymmetries and the relation with the constants of motion it is necessary to enlarge the approach to Killing equations (14) and (15). In [11] it was shown that it is possible to make a weaker demand that an extra supersymmetry exists. It was shown that supersymmetries depend on the existence of a second-rank field $f_{\mu\nu}$ called an *f-symbol*. The f-symbols satisfy relation (15), but the covariant tensor fields $f_{\mu\nu}$ need not necessarily be antisymmetric.

The symmetric part of an f-symbol is the tensor

$$S_{\mu\nu} = \tfrac{1}{2}(f_{\mu\nu} + f_{\nu\mu}) \tag{19}$$

which is a Stäckel–Killing tensor satisfying Killing equation (14). The antisymmetric part

$$A_{\mu\nu} = \tfrac{1}{2}(f_{\mu\nu} - f_{\nu\mu}) \tag{20}$$

obeys the condition

$$D_\mu A_{\nu\lambda} + D_\nu A_{\mu\lambda} = D_\lambda S_{\mu\nu}. \tag{21}$$

3. Non-existence of f-symbols

Bearing in mind that any extension of the original Taub–NUT space does not admit Killing–Yano tensors, in what follows we are in search of f-symbols. For this purpose we seek for solutions of equation (21) with appropriate Stäckel–Killing tensors in the right-hand side.

Taking into account that the Runge–Lenz vector $\vec{S}$ (10) transforms as a vector under rotations generated by $\vec{J}$ according to (12), and in view of the decomposition (18) for the original Taub–NUT space, we are looking for vector and scalar f-symbols. Consequently, in the right-hand side of equation (21) we must have Stäckel–Killing tensors with the corresponding behaviour under three-dimensional rotations.

First we consider equations (21) for a scalar f-symbol with scalar Stäckel–Killing tensors in the right-hand side. From the Killing vectors (3) we can form a trivial scalar Stäckel-Killing tensor

$$\alpha(R_{0\mu}R_{0\nu}) + \beta(R_{1\mu}R_{1\nu} + R_{2\mu}R_{2\nu} + R_{3\mu}R_{3\nu}) \tag{22}$$

with α and β constants.

From the set of partial differential equations (21) we shall select the following ones:

$$\mu = \nu = r \qquad \lambda = \theta : \qquad \frac{\partial A_{r\theta}}{\partial r} - \left(\frac{1}{r} + \frac{f'}{f}\right) A_{r\theta} = 0 \tag{23}$$

$$\mu = \nu = \theta \qquad \lambda = r : \qquad \frac{\partial A_{r\theta}}{\partial \theta} = -\beta\left(f^2 r^3 + \frac{ff'}{2} r^4\right) \tag{24}$$

$$\mu = \nu = r \qquad \lambda = \chi : \qquad \frac{\partial A_{r\chi}}{\partial r} - \left(\frac{f'}{2f} + \frac{g'}{2g}\right) A_{r\chi} = 0 \tag{25}$$

$$\mu = \nu = \chi \qquad \lambda = r : \qquad \frac{\partial A_{r\chi}}{\partial \chi} = -\frac{gg'}{2}(\alpha + \beta) \tag{26}$$

where a prime denotes a derivative with respect to r.

The integrability condition for the pair of equations (25) and (26) is

$$(\alpha + \beta)\left[(g'^2 + gg'') - gg'\left(\frac{f'}{2f} + \frac{g'}{2g}\right)\right] = 0. \tag{27}$$

But for the functions $f(r)$ and $g(r)$ given by (2) which ensure the Kepler-type symmetry of the extended Taub–NUT space, the above relation can be satisfied only for

$$\alpha + \beta = 0. \tag{28}$$

Now from the integrability condition for the pair of equations (23) and (24) we get

$$\beta\left(f^2 r^3 + \frac{ff'}{2} r^4\right)' - \beta\left(\frac{1}{r} + \frac{f'}{f}\right)\left(f^2 r^3 + \frac{ff'}{2} r^4\right) = 0 \tag{29}$$

which, again, for the form (2) of the function $f(r)$ leads to

$$\beta = 0. \tag{30}$$

Therefore, the right-hand side (22) of equation (21) vanishes in the scalar case. But without a symmetric part, the equations for f-symbols lead precisely to equation (15) for Killing–Yano tensors. Now, in conjunction with the absence of the Killing–Yano tensors on extended Taub–NUT space (with the exception of the original Taub–NUT metric) we conclude that not even scalar f-symbols exist.

For a vectorial f-symbol we must consider for the right-hand side of equation (21) a combination between the component $S_{i\mu\nu}$ of the Runge–Lenz vector (10) and the trivial Stäckel–Killing tensor of the form $R_{0\mu}R_{i\nu} + R_{0\nu}R_{i\mu}$ with $i = 1, 2, 3$. Again a detailed analysis of the integrability conditions for equations (21), similar to that done above, leads to the conclusion that vectorial f-symbols do not exist.

4. Conclusions

In supersymmetric quantum mechanics models with standard supersymmetry, the supercharges Q_a close on Hamiltonian H so that we have $\{Q_a, Q_b\} = 2\delta_{ab}H, a, b = 1, \ldots, N$.

In some cases one can find additional hidden supercharges of the non-standard form [11,12] involving the structure constants of a Lie algebra and perhaps Killing–Yano tensors. The appearance of the Killing–Yano tensors is not surprising since they play a role in the existence of hidden symmetries [4, 13]. The analysis of the f-symbols presented in [11] shows that the Killing–Yano and Stäckel–Killing tensors belong to a larger class of possible structures which generate generalized supersymmetry algebras. Unfortunately, to our knowledge, there is no explicit example of f-symbols in the literature.

The absence of Killing–Yano tensors or eventually f-symbols in extended Taub–NUT geometry, in spite of the existence of hidden symmetries in this class of spaces, is quite troublesome. For example, in the formalism of pseudo-classical spinning point particles using anticommuting Grassmann variables to describe the spin degrees of freedom [11, 14], the Killing–Yano tensors or f-symbols play an essential role in the study of generalized Killing equations. The construction of the constants of motion for spinning particles is severely hampered by the absence of the f-symbols. Moreover, in the absence of these geometrical objects, it is not clear how to compute the spin corrections to the conserved quantities corresponding to hidden symmetries or even if these corrections exist.

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